MR4985149 05C35 05C50

Wang, Bing (PRC-CHZU-SMF); **Chen, Wenwen** (PRC-CHZU-SMF); **Zhang, Ping** (PRC-USST-CSC)

Non-r-partite graphs without complete split subgraphs. (English. English summary)

Discrete Math. **349** (2026), *no.* 3, *Paper No.* 114882, 10 *pp.*

Let $T_{n,r}$ denote the r-partite Turán graph on n vertices, and let $\rho(G)$ denote the spectral radius of the graph G. The spectral version of Turán's theorem due to E. Nosal ["Eigenvalues of graphs", M.S. thesis, Univ. Calgary, 1970, doi:10.11575/PRISM/21350] and V. S. Nikiforov [Linear Algebra Appl. **427** (2007), no. 2-3, 183–189; MR2351351] says that $\rho(G) \le \rho(T_{n,r})$ for any K_{r+1}-free graph G on n vertices, with equality if and only if $G \cong T_{n,r}$. Several variations on this theme have been considered recently in the literature under the umbrella of spectral Turán-type problems.

Given a graph H, let $\mathrm{EX}_{r+1}(n, H)$ (resp., $\mathrm{SPEX}_{r+1}(n, H)$) denote the collection of extremal (resp., spectral extremal) graphs for the class of non-r-partite H-free graphs on n vertices, up to isomorphism. Let $Y_r(n)$ be the graph obtained from the Turán graph $T_{n-1,r}$ by removing an edge u_1u_2 between two of the smallest parts and adding a vertex u_0 that is adjacent to u_1, u_2, and all the vertices in the rest of the parts. It is known from the work of A. E. Brouwer [*Two number theoretic sums*, Math. Centrum, Afdeling Zuivere Wiskunde, ZW 19/74, Math. Centrum, Amsterdam, 1974; MR0345918] that $Y_r(n) \in \mathrm{EX}_{r+1}(n, K_{r+1})$, and from the works of H. Lin, B. Ning and B. Wu [Combin. Probab. Comput. **30** (2021), no. 2, 258–270; MR4225787] and Y. Li and Y. J. Peng [SIAM J. Discrete Math. **37** (2023), no. 4, 2462–2485; MR4656787] that $\mathrm{SPEX}_{r+1}(n, K_{r+1}) = \{Y_r(n)\}$.

In the present paper, the authors extend these results by showing that, for $r \ge 3$, $q \ge 1$ and n sufficiently large, $Y_r(n) \in \mathrm{EX}_{r+1}(n, B_{r,q})$ and $\mathrm{SPEX}_{r+1}(n, B_{r,q}) = \{Y_r(n)\}$, where $B_{r,q}$ is the *generalized book graph* (or *complete split graph*) defined as the join of K_r and qK_1. Note that $B_{r,1} \cong K_{r+1}$. They also show that $\mathrm{EX}_{r+1}(n, K_{r+1}) \subseteq \mathrm{EX}_{r+1}(n, B_{r,q})$ for $r \ge 3$, $q \ge 1$ and n sufficiently large. Since K_{r+1} is a subgraph of $B_{r,q}$, these results imply the result of Li and Peng [op. cit.] for sufficiently large n.

Some minor typos: the statement of Theorem 1.4 is missing the hypothesis that G is non-r-partite, and equation (9) should read

$$\rho(G_\ell) - \rho(G) \ge \mathbf{x}^\top \big(A(G_\ell) - A(G)\big)\mathbf{x} \ge \sum_{i \in [\ell-1]} 2x_{u_i}(x_{v_i} - x_{u'}) \ge 0.$$

Brahadeesh Sankarnarayanan

[**References**]

1. J.A. Bondy, U.S.R. Murty, Graph Theory, Grad. Texts Math., vol. 244, Springer, New York, 2008. MR2368647
2. A.E. Brouwer, Some Lotto numbers from an extension of Turán's theorem, in: Report ZW152, Stichting Mathematisch Centrum, Amsterdam, 1981. MR0611510
3. S. Cioabă, D.N. Desai, M. Tait, The spectral radius of graphs with no odd wheels, Eur. J. Comb. 99 (2022) 103420. MR4308011
4. S. Cioabă, L.H. Feng, M. Tait, X.-D. Zhang, The maximum spectral radius of graphs without friendship subgraphs, Electron. J. Comb. 27 (4) (2020) 4.22. MR4245197
5. D.N. Desai, L.Y. Kang, Y.T. Li, Z.Y. Ni, M. Tait, J. Wang, Spectral extremal graphs for intersecting cliques, Linear Algebra Appl. 644 (2022) 234–258. MR4402010
6. P. Erdős, Some recent results on extremal problems in graph theory (results), in: Theory of Graphs (International Symposium, Rome, 1966, Gordon and Breach, New

York, Dunod, Paris, 1966, pp. 117–123. MR0227049
7. P. Erdős, On some new inequalities concerning extremal properties of graphs, in: Theory of Graphs (Proceedings of the Colloquium, Tihany, 1966, Academic Press, New York, 1968, pp. 77–81. MR0232703
8. B.D. Guiduli, Spectral extrema for graphs, Ph.D. Thesis, 105 pages, University of Chicago, December 1996. MR2716665
9. Y.T. Li, Y.J. Peng, Refinement of spectral Turán's theorem, SIAM J. Discrete Math. 37 (4) (2023) 2462–2485. MR4656787
10. Y.T. Li, Y.J. Peng, The spectral radius of graphs with no intersecting odd cycles, Discrete Math. 345 (8) (2022) 112907. MR4401924
11. Y.T. Li, Y.J. Peng, The maximum spectral radius of non-bipartite graphs forbidding short odd cycles, Electron. J. Comb. 29 (4) (2022) 4.2. MR4497214
12. H.Q. Lin, B. Ning, B. Wu, Eigenvalues and triangles in graphs, Comb. Probab. Comput. 30 (2) (2021) 258–270. MR4225787
13. W. Mantel, Problem 28, solution by H. Gouwentak, W. Mantel, J. Teixeira de Mattes, F. Schuh and W.A. Wythoff, Wiskundige Opgaven 10 (1907) 60–61.
14. V. Nikiforov, Bounds on graph eigenvalues II, Linear Algebra Appl. 427 (2–3) (2007) 183–189. MR2351351
15. V. Nikiforov, Some new results in extremal graph theory, in: Surveys in Combinatorics 2011, in: London Math. Soc. Lecture Note Ser., vol. 392, Cambridge Univ. Press, Cambridge, 2011, pp. 141–181. MR2866733
16. V. Nikiforov, Stability for large forbidden subgraphs, J. Graph Theory 62 (4) (2009) 362–368. MR2561002
17. V. Nikiforov, Spectral saturation: inverting the spectral Turán theorem, Electron. J. Comb. 16 (1) (2009) 33. MR2491635
18. V. Nikiforov, The spectral radius of graphs without paths and cycles of specified length, Linear Algebra Appl. 432 (9) (2010) 2243–2256. MR2599857
19. E. Nosal, Eigenvalues of Graphs, Master's thesis, University of Calgary, 1970.
20. M. Simonovits, A method for solving extremal problems in graph theory, stability problems, in: Theory of Graphs, Proc. Colloq., Tihany, 1966, Academic Press, New York, 1968, pp. 279–319. MR0233735
21. P. Turán, On an extremal problem in graph theory, Mat. Fiz. Lapok 48 (1941) 436–452. MR0018405
22. J. Wang, L.Y. Kang, Y.S. Xue, On a conjecture of spectral extremal problems, J. Comb. Theory, Ser. B 159 (2023) 20–41. MR4519328
23. H. Wilf, Spectral bounds for the clique and indendence numbers of graphs, J. Comb. Theory, Ser. B 40 (1986) 113–117. MR0830598
24. B.F. Wu, E.L. Xiao, Y. Hong, The spectral radius of trees on k pendant vertices, Linear Algebra Appl. 395 (2005) 343–349. MR2112895
25. M.Q. Zhai, H.Q. Lin, A strengthening of the spectral chromatic critical edge theorem: books and theta graphs, J. Graph Theory 102 (3) (2023) 502–520. MR4563204
26. M.Q. Zhai, R.F. Liu, J. Xue, A unique characterization of spectral extrema for friendship graphs, Electron. J. Comb. 29 (2022) 3. MR4467134
27. A.A. Zykov, On some properties of linear complexes, Mat. Sb. (N.S.) 24 (1949) 163–188 (in Russian). MR0035428

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.